Momentum and Heat Transfer in Laminar Boundary-Layer Flows of Non-Newtonian Fluids Past External Surfaces

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A theoretical analysis for the laminar flow past arbitrary external surfaces of non-Newtonian fluids of the power-law model is presented. The main problem which is considered is how to predict the drag and the rate of heat transfer from an isothermal surface to the fluid. Inspectional analysis of the modified boundary-layer equations yields a general relationship both for the drag coefficient and for the Nusselt number as functions of the generalized Reynolds and Prandtl numbers. The flow past a horizontal flat plate is studied in detail numerically.

Considerable attention has lately been devoted to the problem of how to predict the behavior of non-Newtonian fluids in motion. The main reason for this is probably that fluids, such as molten plastics, pulps, slurries, emulsions, etc., which do not obey the Newtonian postulate that the stress tensor is directly proportional to the deformation tensor, are produced industrially in increasing quantities and are therefore in some cases just as likely to be pumped in a plant as the more common Newtonian fluids.

It is well known, though, that a chief difficulty in the theoretical study of non-Newtonian fluid mechanical phenomena, and in the correct interpretation of experimental results, is that so far no definite relationship between the stress tensor and the deformation tensor, valid for all fluids, has been discovered. This means that, except for simple cases, a generalized form of the Navier-Stokes equations, obeyed by all fluids in motion, cannot be written down, and so theoretical studies in this area are limited not so much by the mathematical complexity of the basic equations as by the inability to arrive at their correct form. It is fortunate however that certain simple problems in this field can be attacked successfully, either theoretically or experimentally, and thereby provide a basis for the analysis of the more complicated physical situations usually encountered in practice.

The flow in pipes and channels has been primarily studied so far because of its importance and relative simplicity. The laminar-flow equations have been solved (*4, 10, 11*) for various non-Newtonian models, characterized by the empirically observed relationship between shear stress and velocity gradient, and a generalization of the Poiseuille formula for pressure drop has been derived. In addition, some experimental work has been carried out in the turbulent regime for fluids which obey the power-law model (*11, 13*), and a tentative extension of the familiar Fanning friction-factor plot has been proposed (*2, 11, 13, 16*). The analogous laminar heat transfer problem has also been studied (*5, 9, 10*), and generalized j factor plots for power-law fluids have been made available (*12*).

It is thus seen that the flow inside pipes of non-Newtonian fluids, especially those obeying the power-law model, has been covered both theoretically and experimentally rather well, although it is realized that further work in the turbulent regime would be both desirable and most useful. Other investigations have been restricted to rather specialized problems, for example the design of an extruder for a pseudoplastic fluid (*1*).

It is surprising to note however that, according to the authors' best knowledge, no analysis of the flow of non-Newtonian fluids past external surfaces has ever been published. Such systems have, aside from their practical applications, considerable theoretical interest, for, especially when the geometry of the external surface is simple, they can be examined more carefully and in more detail than internal flows and thus yield fundamental information about the behavior of non-Newtonian fluids in motion.

In the present paper a theoretical analysis will be presented of the laminar flow of non-Newtonian fluids which obey the power-law model past an arbitrary two-dimensional surface. The main problem considered is that of predicting the drag and the rate of heat transfer from an isothermal surface to the fluid. Inspectional analysis of the modified boundary-layer equations yields a general relationship both for the drag coefficient and for the Nusselt number as functions of the generalized Reynolds and Prandtl numbers. Also included is a rather detailed numerical study of the flow past a horizontal flat plate.

BASIC LAMINAR BOUNDARY-LAYER EQUATIONS AND THEIR INSPECTIONAL ANALYSIS

The laminar flow of a non-Newtonian fluid past the arbitrary two-dimensional surface, shown in Figure 1, is considered. Analysis will be restricted to the case where the usual boundary-layer assumptions can be made that gradients in the normal direction are much larger numerically than the corresponding gradients in the transverse, or x, direction. This allows simplifying the exact basic equations of conservation of momentum, mass, and energy and transforming them into the boundary-layer equations. For a constant property fluid and in the absence of the dissipation function in the energy equation, which may usually be neglected to a first approximation, these are *(14)*

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_i\frac{dU_i}{dx} + \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} \qquad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} \qquad (3)$$

The symbols have the usual meaning. The boundary conditions are

$$\begin{array}{l} \text{At } y = 0,\ u = 0,\ v = 0\ T = T_s \\ \text{At } y = \infty,\ u = U_i(x),\ T = T_\infty \\ \text{At } x = 0,\ u = U_i(o),\ T = T_\infty \end{array} \qquad (4)$$

It now remains to express τ_{xy} in terms of the velocity gradients by means of some empirical equation of state. It was pointed out in the introduction that a generally acceptable definite relationship between the components of the stress tensor and those of the deformation tensor has not yet been proposed. In accordance with the boundary-layer theory, however, $\partial u/\partial y$ is numerically much larger than all the other velocity gradients. It follows therefore that for boundary-layer flows τ_{xy} is a function of $\partial u/\partial y$ only, which, as is well known, is exact for the fully developed flow in pipes and channels where all the velocity gradients, except for $\partial u/\partial y$, are identically zero.

Generally the form of the function relating τ_{xy} to $\partial u/\partial y$ is quite complicated. It has been found however that the two-parameter equation of state

$$\tau_{xy} = K\left(\frac{\partial u}{\partial y}\right)^n \qquad (5)$$

is adequate for many non-Newtonian fluids *(10)*. This is the well-known power-law model which will be used in the remaining part of this paper.

Equations (1 to 5) possess now a unique solution that will depend on the surface geometry, which, as is well known from potential theory, establishes the function $U_i(x)$, as well as on the characteristic parameters of the fluid, K, n, ρ, c_p and k. First, however, important and general results can be obtained, without solving the equations, by means of inspectional analysis. This is sometimes referred to in the literature as the stretching of the boundary-layer coordinates. If the following transformations are made,

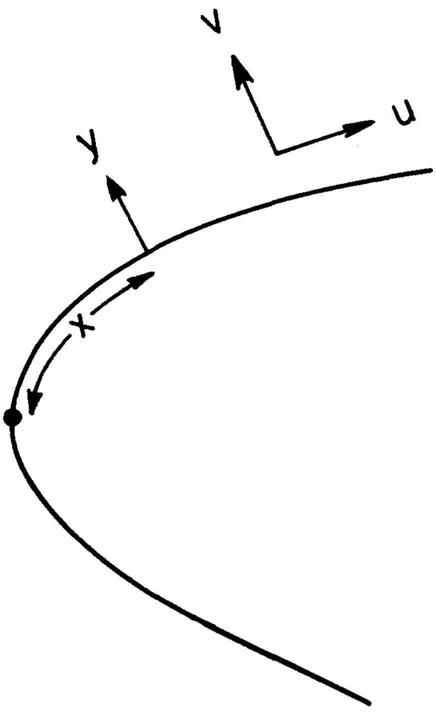

Fig. 1. The directions of the coordinates x and y and the velocities u and v with respect to the surface.

$$x_1 \equiv \frac{x}{L};\ u_1 \equiv \frac{u}{U_\infty};\ \phi(x_1) \equiv \frac{U_i(x_1)}{U_\infty};$$

$$\theta \equiv \frac{T - T_\infty}{T_s - T_\infty};\ y_1 \equiv \left[\frac{\rho U^{2-n}_\infty L^n}{K}\right]^{\frac{1}{1+n}}\frac{y}{L};$$

$$v_1 \equiv \left[\frac{\rho U^{2-n}_\infty L^n}{K}\right]^{\frac{1}{1+n}}\frac{v}{U_\infty};$$

$$N_{Re} \equiv \left[\frac{\rho U^{2-n}_\infty L^n}{K}\right];$$

$$\text{and } N_{Pr} \equiv \frac{N_{Pe}}{N_{Re}^{2/(1+n)}} \equiv \frac{c_p U_\infty \rho L}{k N_{Re}^{2/(1+n)}} \qquad (6)$$

the boundary-layer equations become

$$u_1\frac{\partial u_1}{\partial x_1} + v_1\frac{\partial u_1}{\partial y_1} = \phi\frac{d\phi}{dx_1} + \frac{\partial}{\partial y_1}\left(\frac{\partial u_1}{\partial y_1}\right)^n \qquad (7)$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} = 0 \qquad (8)$$

$$u_1\frac{\partial \theta}{\partial x_1} + v_1\frac{\partial \theta}{\partial y_1} = \frac{1}{N_{Pr}}\frac{\partial^2\theta}{\partial y_1^2} \qquad (9)$$

while the boundary conditions are changed into

$$\begin{array}{lll} \text{At } y_1 = 0,\ u_1 = 0, & v_1 = 0, & \theta = 1 \\ \text{At } y_1 = \infty & u_1 = \phi(x_1), & \theta = 0 \\ \text{At } x = 0 & u_1 = \phi(0), & \theta = 0 \end{array} \qquad (10)$$

It therefore follows immediately that if τ_o is the shear stress at the wall and the Nusselt number is defined in the usual manner as

$$N_{Nu} \equiv -L\left(\frac{\partial\theta}{\partial y}\right)_{y=0}$$

then

$$\frac{\tau_o}{\rho U_\infty^2} = (N_{Re})^{-\frac{1}{1+n}}\left(\frac{\partial u_1}{\partial y_1}\right)^n_{y=0} = (N_{Re})^{-\frac{1}{1+n}}F_1(x_1, n) \qquad (11)$$

and

$$N_{Nu} = (N_{Re})^{\frac{1}{1+n}}\left(\frac{\partial\theta}{\partial y_1}\right)_{y_1=0} = (N_{Re})^{\frac{1}{1+n}}F_2(x_1, N_{Pr}, n) \qquad (12)$$

In many instances, however, one is interested in the average values of τ_o and N_{Nu}, which are calculated from Equations (11) and (12) by integrating over that part of the surface where the boundary-layer equations apply. Thus for non-Newtonian fluids of the power law type

$$\frac{\overline{\tau_o}}{\rho U_\infty^2} = (N_{Re})^{-\frac{1}{1+n}}F_3(n) \qquad (13)$$

and

$$\overline{N_{Nu}} = (N_{Re})^{\frac{1}{1+n}}F_4(N_{Pr}, n) \qquad (14)$$

where $F_3(n)$ depends on n and $F_4(N_{Pr}, n)$ is also a function of the Prandtl number which, as will be shown later on, can be simplified still further.

It should also be noted here that Equations (13) and (14) are themselves of considerable interest, for they prove that in laminar boundary-layer flows the two groups $[\overline{\tau_o}/\rho U_\infty^2](N_{Re})^{1/(1+n)}$ and $\overline{N_{Nu}}(N_{Re})^{-1/(1+n)}$ are both independent of the Reynolds number, since F_3 and F_4 depend only on the geometry of the surface and in the case of F_4 the Prandtl number. Therefore when the boundary-layer equations are too involved and cannot be solved, Equations (13) and (14) can be used for the analysis and the correlation of experimental data. It also should be quite evident that all the results presented so far reduce to the corresponding expressions for Newtonian fluids when $n = 1$.

LIMITATIONS OF THE BOUNDARY-LAYER THEORY

It must be remembered that the boundary-layer equations are not exact but are the asymptotic forms of the basic hydrodynamic relations when the Reynolds number is large. It would be of interest to examine under what conditions laminar boundary-layer type of flows would be expected to occur.

It has already been seen that the characteristic Reynolds number for a power law non-Newtonian fluid is

$$N_{Re} = \frac{\rho U_\infty^{2-n} L^n}{K}$$

from which it follows that only if $n < 2$ is N_{Re} a monotonic increasing function of U_∞. It is apparent therefore that for $n < 2$ laminar boundary-layer flows may be produced by making U_∞ sufficiently large. For $n > 2$ however a boundary-layer configuration will not be formed if U_∞ is too large, since N_{Re} decreases with increasing U_∞. And although the above expression predicts that for $n > 2$, $N_{Re} \to \infty$ as $U_\infty \to 0$, boundary-layer flows do not occur when the characteristic velocity is small because the power-law model

$$\tau_{xy} = K\left(\frac{\partial u}{\partial y}\right)^n$$

is valid only when the velocity gradient component $\partial u/\partial y$ is relatively large (Figure 3 reference *10*). Thus when U_∞, and therefore $\partial u/\partial y$, is small, then, as with Newtonian fluids, τ_{xy} must be a linear function of $[(\partial u)/(\partial y) + (\partial v)/(\partial x)]$ for two-dimensional flows, since on physical grounds τ_{xy} is analytical and for sufficiently small values of $\partial u/\partial y$ and $\partial v/\partial x$ can be expressed by a power series in these variables. One can conclude from this discussion then that:

1. All fluids approach Newtonian behavior if U_∞ is sufficiently small and, in accordance with the well-known argument of Stokes, the inertia terms of the equations of motion may be neglected.
2. For $n < 2$ boundary-layer type of flows can be obtained if U_∞ is large, and therefore the Reynolds number is made sufficiently large.
3. For $n > 2$ and moderate values of U_∞ the boundary-layer approximations may be introduced provided that $N_{Re} >> 1$. For large values of U_∞ the inertia terms may again be neglected since $N_{Re} \to 0$; however the stress-strain-velocity relations would have to be made more elaborate than the simple power-law form.

It is evident therefore than when $n > 2$ the boundary-layer flow is not an asymptotic state of laminar motion which is approached as U_∞ is made sufficiently large. At best there may be an intermediate state of high Reynolds number, where the boundary layer approximations are valid, which lies between regions characterized by the fact that the inertia terms in the equations of motion may be neglected. It follows then that when $n > 2$ laminar boundary-layer flows are probably not of much practical interest because their range of validity appears to be rather limited.

THE FLOW PAST A HORIZONTAL FLAT PLATE

The analysis of the horizontal flat plate will be considered now in some detail. It is divided into two parts: the solution of Equations (7) and (8) which will yield the velocity distribution and the local value of the shear stress at the wall, and the determination of the local rate of heat transfer from the surface to the main part of the fluid.

The velocity distribution and the drag on a flat plate

For the flow past a flat plate $\phi = 1$, and a Blasius type of similarity transformation can be used to reduce Equations (7) and (8) into an ordinary differential equation. Thus if

$$\eta \equiv \frac{y_1}{x_1^{\frac{1}{1+n}}} \text{ and } u_1 \equiv f'(\eta) \qquad (15)$$

it can be shown, from Equations (7) and (8), that

$$v_1 = \frac{1}{n+1} x_1^{-\frac{n}{n+1}} [\eta f' - f] \qquad (15a)$$

and

$$n(n+1)f''' + (f'')^{2-n} f = 0 \qquad (16)$$

Incidentally an analogous transformation can also be used for the flow past a wedge, for which $\phi = x_1^m$.

Equation (16), which for $n = 1$ will be recognized as the familiar Blasius equation (*14*), is nonlinear in f except when $n = 2$. It must be solved numerically, subject to the usual boundary conditions:

$$f = f' = 0 \quad \text{at} \quad \eta = 0$$
$$f' = 1 \quad \text{at} \quad \eta = \infty$$

Also by the use of a technique first introduced by Töpfer (*15*) in connection with the Blasius equation, no trial and error is required. One can easily see however that when $n \geqq 2$ a function which will satisfy both Equation (16) and the above conditions cannot be found. Therefore the boundary conditions must be changed into the more general form

$$f = f' = 0 \text{ at } \eta = 0$$
$$f' = 1 \text{ and } f'' = 0 \text{ for } \eta \geqq \eta_c \qquad (17)$$

Of course $\eta_c = \infty$ for $n < 2$, but η_c is finite for $n \geqq 2$. This interesting phenomenon of a finite boundary-layer thickness is never encountered in the laminar flow of Newtonian fluids. It was pointed out however that when $n > 2$, laminar boundary-layer flows are probably not of much practical interest, since their range of validity appears to be rather limited; therefore the numerical calculations for $n > 2$, which are included here for completeness, seem to be of secondary significance only. On the other hand Equation (16) simplifies as $n \to 0$, for if

$$f'(\eta) \equiv f'_1(\eta/n) \qquad (18)$$

f_1 becomes independent of n and is obtained from

$$f'''_1 + [f''_1]^2 f_1 = 0 \qquad (18a)$$

This limiting expression is accurate when $n < 0.1$ in the sense that both Equations (16) and (18*a*) yield essentially identical solutions.

Velocity profiles, calculated from Equations (16) and (17), are seen plotted in Figures 2 and 3 for dilatant ($n > 1$) and pseudoplastic fluids ($n < 1$). Generally however it is the shear stress at the wall and not the complete velocity profile which is of prime interest. From Equations (11) and (15)

$$\frac{\tau_o}{\rho U_\infty^2} = (N_{Re})^{-\frac{1}{1+n}} \frac{[f''(o)]^n}{x_1^{\frac{n}{1+n}}}$$

$$\equiv c(n)\,(N_{Re_x})^{-\frac{1}{1+n}}$$

where $c(n) \equiv [f''(o)]^n$ and $N_{Re_x} \equiv (\rho U_\infty^{2-n} x^n)/K$ (a Reynolds number based on x).

The shear-stress coefficient can be determined exactly from the numerical solution of Equation (18) or approximately by a momentum integral method, analogous to the one developed by Pohlhausen for Newtonian fluids (*14*). Owing to its relative simplicity and good accuracy this Pohlhausen method is very frequently used in boundary-layer problems, where the exact solution of the basic equations would be too involved and time consuming to be carried out. It is therefore of considerable interest to compare the results of the integral method and those of the exact numerical solution for the flow past a flat plate. This is done to obtain some feeling as to the accuracy of the approximate procedure and as to whether or not it could be used with the same degree of confidence for non-Newtonian as well as Newtonian fluids.

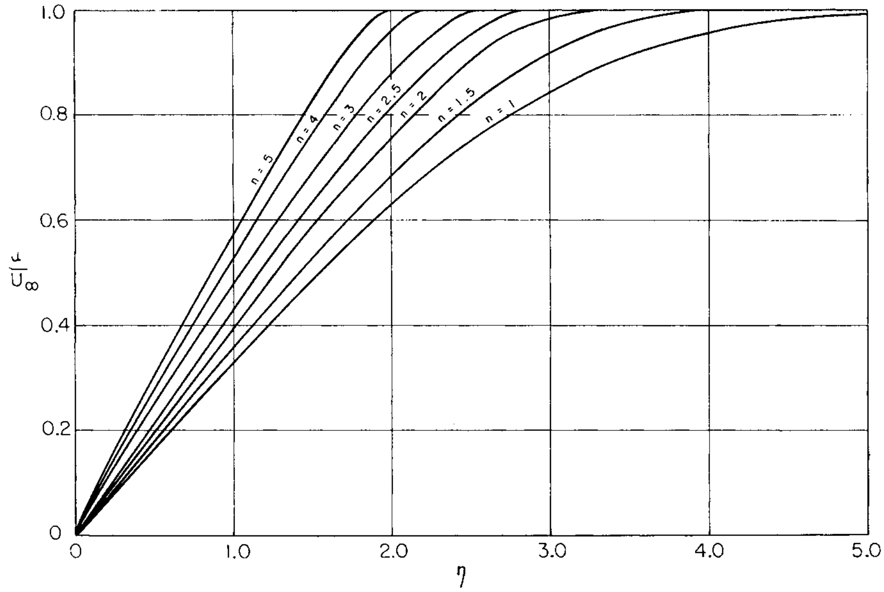

Fig. 2. Velocity profile over a flat plate for dialatant fluids: $\tau_{xy} = K\left(\frac{\partial u}{\partial y}\right)^n$**, $n \geqq 1$, η given by Equation (15).**

The approximate momentum-integral method

If Equations (7) and (8), with $\phi = 1$ for flat plate, are multiplied through by dy_1 and integrated from 0 to ∞, they reduce to

$$\left(\frac{\partial u}{\partial y_1}\right)^n_{y_1=0} = \frac{d}{dx_1}\int_0^\infty (1-u_1)\, u_1\, dy_1 \quad (20)$$

which will be solved now by the Pohlhausen method (*14*, p. 201).

A commonly used profile, for the flow past a flat plate, is

$$u_1 = \frac{3y_1}{2\delta} - \frac{y_1^3}{2\delta^3} \text{ for } y_1 \leqq \delta$$
$$u_1 = 1 \text{ for } y_1 \geqq \delta \quad (21)$$

This gives, when substituted into Equation (20),

TABLE 1. COMPARISON BETWEEN THE EXACT VALUES OF $c(n)$ AND EQUATION (22)

n	$c(n)$ exact	$c(n)$ from Equation (22)
0.05	1.017	0.926
0.1	0.969	0.860
0.2	0.8725	0.75
0.3	0.7325	0.655
0.5	0.5755	0.518
1.0	0.33206	0.323
1.5	0.2189	0.238
2.0	0.1612	0.169
2.5	0.1226	0.133
3.0	0.09706	0.109
4.0	0.06777	0.079
5.0	0.05111	0.061

$$\delta^{n+1} = \frac{280}{39}(n+1)\left(\frac{3}{2}\right)^n x_1$$

and therefore from Equations (11) and (19)

$$\frac{\tau_o}{\rho U_\infty^2} = c(n)\,(N_{Re_x})^{-\frac{1}{1+n}}$$

where

$$c(n) \equiv \left[\frac{39}{280}\cdot\frac{1.5}{n+1}\right]^{\frac{n}{n+1}} \quad (22)$$

A comparison between the exact values of $c(n)$ and Equation (22) is shown in Figure 4 and Table 1.

Similarly the profile

$$u_1 = \sin\frac{\pi y_1}{2\delta} \text{ for } y_1 \leqq \delta$$

gave results very close to those of Equation (22). However the comparison between the numerically calculated values of $c(n)$ and those obtained from the momentum integral method shows that the latter is of acceptable accuracy when $n \leqq 3$ and that the best results can be expected where n is approximately 1. Moreover from experience with this method for Newtonian fluids one should also expect that the inaccuracy of the Pohlhausen method will be larger for more complicated surfaces and that still poorer results would be obtained if this method were used to predict the location of the separation point. For it is known that even for Newtonian fluids the separation point cannot be calculated accurately by means of integral methods.

Table 1 will also show that the modified Pohlhausen method is unable, when n is small, to predict the numerical value of $f''(o)$ with reasonable accuracy. This must be considered as a serious disadvantage of the momentum-integral approach because, as will be brought out in the next section, an accurate knowledge of $(\partial u/\partial y)_{y=0}$ is essential for most heat transfer calculations involving non-Newtonian fluids. It can be stated in conclusion therefore that the Pohlhausen method appears to be less accurate in general for non-Newtonian than it is for Newtonian fluids, but for $n < 3$ it could be used to predict τ_o, although not $(\partial u/\partial y)_{y=0}$, with acceptable accuracy. It seems probable in addition that this integral method would turn out to be quite

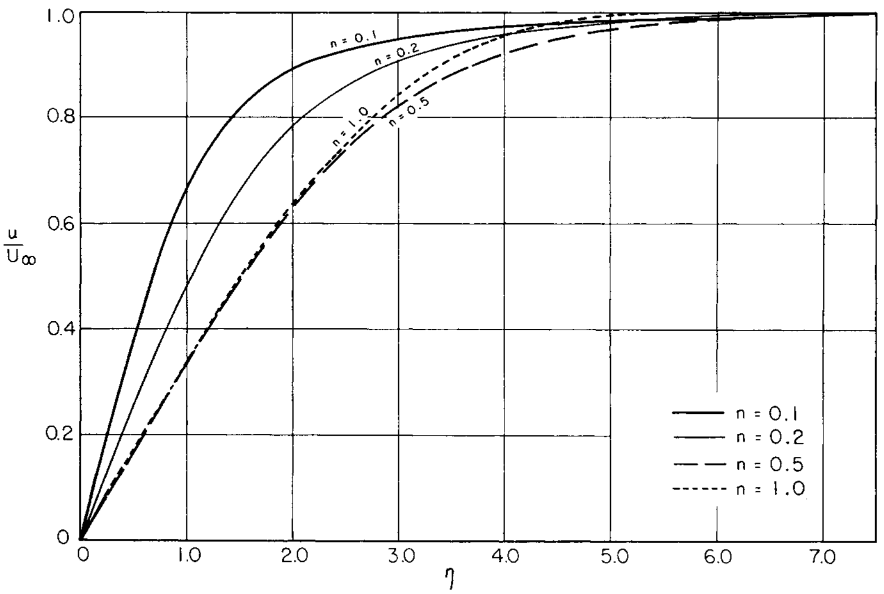

Fig. 3. Velocity profile over a flat plate for pseudoplastic fluids: $\tau_{xy} = K\left(\frac{\partial u}{\partial y}\right)^n$**, $0 < n \leqq 1$, η given by Equation (15).**

unreliable for the calculation of the separation point.

THE RATE OF HEAT TRANSFER FROM AN ISOTHERMAL SURFACE

Once the velocity distribution past a surface has been determined, u_1 and v_1 can be substituted in Equation (9). The solution of this linear equation will then yield the rate of heat transfer from the surface, and also the temperature profile, which usually however is of little interest. Equation (9) cannot generally be solved by elementary means, and an exact solution can at present be obtained only by a numerical finite-difference procedure. Most such problems for Newtonian fluids are therefore solved approximately by a standard momentum-integral method (*6*, *14*) similar to that of Pohlhausen, which was briefly described earlier. This method can also be applied, without any modifications, to non-Newtonian fluids because, as is obvious, Equation (9) is perfectly general and not restricted to any particular class of fluids. On the other hand the local rate of heat transfer can also be calculated by means of a closed-form expression first obtained by Lighthill (*8*). Strictly speaking it is exact only for very large Prandtl numbers, but, as Lighthill has shown for Newtonian fluids, it can be used with good accuracy for $N_{Pr} \geqq 1$. This formula is apparently not well known to chemical engineers, and therefore a derivation of it is presented below which is simpler than that given originally by Lighthill. As was first pointed out by Fage and Falkner (*3*) the temperature boundary layer is much thinner over a given surface than the momentum boundary layer for large Prandtl numbers. Therefore essentially all the temperature drop occurs in a very narrow region close to the wall, where the velocity u_1 is given by the first term of a power series expansion in y_1:

$$u_1 = \beta(x_1)\, y_1 \tag{23}$$

This is then substituted in Equation (9) which can readily be transformed into

$$\beta y_2 \frac{\partial \theta}{\partial x_1} - \frac{\beta' y_2^2}{2} \frac{\partial \theta}{\partial y_2} = \frac{\partial^2 \theta}{\partial y_2^2} \tag{24}$$

where

$$y_2 \equiv (N_{Pr})^{1/3}\, y_1 \tag{24a}$$

It can be shown now that Equation (24) can be reduced to an ordinary differential equation by a similarity transformation. Thus if θ equals a function of z only, where

$$z \equiv \frac{\beta(x_1)}{2t^{2/3}}\, y_2^2,\quad t\,(x_1) = \int_0^{x_1} \sqrt{2\beta}\, dx_1 \tag{25}$$

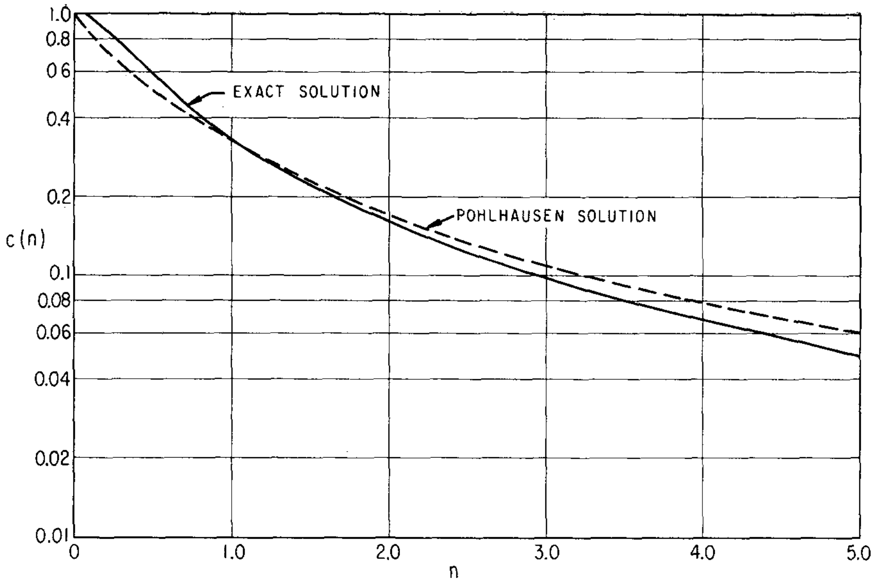

Fig. 4. A comparison between the exact and the approximate solution for *c(n)*.

then Equation (24) becomes

$$\frac{d^2\theta}{dz^2} + \left(\frac{1}{2z} + \frac{2z^{1/2}}{3} \right) \frac{d\theta}{dz} = 0 \tag{26}$$

The solution, which satisfies the boundary conditions

$$\theta = 0 \text{ at } z = \infty, \quad \theta = 1 \text{ at } z = 0$$

is

$$\theta = \frac{1}{\Gamma(1/3)} \int_{\frac{4z^{3/2}}{9}}^{\infty} x^{-2/3} e^{-x}\, dx \tag{27}$$

It follows that

$$-\left(\frac{\partial \theta}{\partial y_1} \right)_{y_1=0} = \frac{1}{0.8930} \left(\frac{N_{Pr}}{9} \right)^{1/3} \frac{\sqrt{\beta(x_1)}}{\left[\int_0^{x_1} \sqrt{\beta}\, dx_1 \right]^{1/3}} \tag{28}$$

which is the Lighthill formula. For large Prandtl numbers therefore Equation (14) can be simplified to

$$\overline{N_{Nu}} = (N_{Re})^{\frac{1}{1+n}} (N_{Pr})^{1/3} F_5(n) \tag{29}$$

where $F_5(n)$ is a function only of n and the surface geometry. It is seen then that for non-Newtonian fluids as well the average Nusselt number is again asymptotically proportional to the 1/3 power of the Prandtl number, although it should be kept in mind that the Lighthill formula should not be expected to hold near the separation point, where $\beta = 0$.

The rate of heat transfer from a horizontal flat plate

By means of Equation (28) an expression for the rate of heat transfer from a horizontal flat plate will now be obtained. One can readily show from Equations (15) and (19) that

$$\beta(x_1) = [c(n)]^{1/n}\, x_1^{-\frac{1}{1+n}} \tag{30}$$

where $c(n)$ is given in Figure 4 and Table 1. Therefore in accordance with Equation (28)

$$-\left(\frac{\partial \theta}{\partial y} \right)_{y_1=0} = \frac{1}{0.8930} \left[\frac{c(n)^{1/n} N_{Pr}}{9} \frac{1+2n}{2+2n} \right]^{1/3} x_1^{-\frac{2+n}{3(1+n)}} \tag{31}$$

so that the local value of the Nusselt number based on the distance from the leading edge is

$$N_{Nu_x} \equiv -x \left(\frac{\partial \theta}{\partial y} \right)_{y=0} = \frac{(N_{Re})^{\frac{1}{1+n}}}{0.8930} \left[\frac{c(n)^{1/n} N_{Pr}}{9} \frac{1+2n}{2+2n} \right]^{1/3} x_1^{\frac{1+2n}{3(1+n)}} \tag{32}$$

It remains now to determine whether Equation (28) is applicable along the whole surface of the flat plate or only along part of it. One finds from Equation (15) that the momentum boundary-layer thickness is proportional to $x_1^{1/(1+n)}$ and from Equation (25) that the temperature boundary-layer thickness is proportional to $x_1^{\frac{2+n}{3(1+n)}}$. Therefore the ratio of the temperature to the momentum boundary-layer thickness is proportional to $x_1^{\frac{n-1}{3(1+n)}}$.

For Equation (28) to be accurate, this ratio must be very small, so that for $n < 1$, Equation (31) is valid as $x_1 \to \infty$; for $n > 1$, Equation (31) is valid as $x_1 \to 0$.

It can be shown now that for $n < 1$ there exists a region near the leading edge where the momentum boundary-layer thickness is much smaller than that of the temperature. When one calculates the rate of heat transfer for $x_1 \to 0$ and $n < 1$, therefore, the velocity distribution in the boundary may be disregarded. Thus by setting $u_1 = 1$ and $v_1 = 0$ in Equation (9) and solving the resulting expression by elementary means one finds that

$$-\left(\frac{\partial \theta}{\partial y_1}\right)_{y=0} = \frac{1}{\sqrt{\pi}} x_1^{-1/2} (N_{Pr})^{1/2} \tag{33}$$

and

$$N_{Nu_x} = \frac{x_1^{1/2}}{\sqrt{\pi}} (N_{Re})^{\frac{1}{1+n}} (N_{Pr})^{1/2} \tag{33a}$$

Two asymptotic formulas, Equations (32) and (33*a*) have been derived then for N_{Nu_x}. It has been found from experience however that for such heat transfer problems it is usually sufficiently accurate to set N_{Nu_x} equal to either Equation (32) or (33*a*), whichever is appropriate. Let therefore x_1^* be defined as the intersection of these two asymptotes, so that

$$\frac{(x_1^*)^{1/2}}{\sqrt{\pi}} (N_{Re})^{\frac{1}{1+n}} (N_{Pr})^{1/2} = \frac{(N_{Re})^{\frac{1}{1+n}}}{0.8930} \left[\frac{c(n)^{1/n} N_{Pr}}{9} \, \frac{1+2n}{2+2n}\right]^{1/3} x_1^{*\frac{1+2n}{3(1+n)}}$$

and

$$(x_1^*)^{\frac{n-1}{6(1+n)}} = 1.3157 \, (N_{Pr})^{1/6} \left[\frac{2+2n}{1+2n}\left(\frac{1}{c(n)}\right)^{1/n}\right]^{1/3} \tag{34}$$

With good accuracy then

1. For $n < 1$, N_{Nu_x} is given by Equation (33*a*) for $0 \leqq x_1 \leqq x_1^*$ and by Equation (32) for $x_1 \geqq x_1^*$.

2. For $n > 1$, N_{Nu_x} is obtained from Equation (32) for $0 \leqq x_1 \leqq x_1^*$ and from Equation (33*a*) for $x_1 \geqq x_1^*$.

One observes however that as $N_{Pr} \to \infty$, $x_1^* \to 0$ for $n < 1$, and $x_1^* \to \infty$ for $n > 1$; therefore for large Prandtl numbers Equation (28) is applicable essentially over the whole surface of the flat plate. This is very convenient, since the Prandtl number is usually quite large, as will be shown by the following numerical example.

In the flow of a lime-water suspension, the rheological properties of which are (*10*) $n = 0.17$, $K = 5.48$ lb.-mass sec.$^{n-2}$/ft., $\rho = 90$ lb.-mass cu. ft., $c_p = 0.7$ B.t.u./lb.-mass °F., and $k = 0.7$ B.t.u./ft. °F. hr. $U_\infty = 18{,}000$ ft./hr., $L = 1$ ft., $c(n)^{1/n} = f''(0) = 0.504$. Therefore $N_{Re} = 312$, $N_{Pr} = 86.2$, and from Equation (34) $x_1^* \sim 10^{-5}$.

Since Equation (28) is general, these results can be generalized to arbitrary surfaces, provided that the function $\beta(x_1)$ can be obtained first from the solution of the equations of motion. Since for most problems involving non-Newtonian fluids N_{Pr} is large, the Lighthill formula, Equation (28), can be used for calculating the local rate of heat transfer over that part of a surface where the boundary-layer equations apply. It is true that, as was shown above, one should expect Equation (28) to break down over a portion of the surface where the momentum boundary-layer thickness is smaller than that of the temperature. This is especially true when $n \to 0$, as is clear from Equation (18). Under these conditions the velocity distribution in the boundary layer can be overlooked, and the fluid may be assumed ideal for the purpose of obtaining the rate of heat transfer with standard procedures (7). In general however the Prandtl number is quite large, and this region is so small that its existence can for all practical purposes be disregarded.

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NOTATION

- $c(n)$ = shear-stress coefficient defined by Equation (19)
- c_p = specific heat
- f = defined by Equation (15)
- F_1, F_2 = functions of indicated variables, which have to be obtained from the solution of Equations (7) to (9)
- k = thermal conductivity
- K, n = parameters in the power law model, Equation (5)
- L = characteristic length
- N_{Nu} = Nusselt number
- N_{Pr} = $(c_p U_\infty \rho L)/k(N_{Re})^{2/(1+n)}$, the Prandtl number
- N_{Re} = $(\rho U_\infty^{2-n} L^n)/K$, the Reynolds number
- N_{Pe} = conventional Peclet number
- T = temperature
- T_s = temperature of the surface
- T_∞ = temperature of the bulk of the fluid
- $U_i(x)$ = the velocity component outside the boundary layer
- U_∞ = a characteristic velocity
- u = velocity component along x
- u_1 = dimensionless velocity component defined by Equation (6)
- v = velocity component along y
- v_1 = dimensionless velocity component defined by Equation (6)
- x = distance along the surface from the leading edge
- x_1 = dimensionless distance defined by Equation (6)
- x_1^* = defined by Equation (34)
- y = distance normal to the surface
- y_1 = dimensionless distance defined by Equation (6)

Greek Letters

- $\beta(x_1)$ = defined by Equation (23)
- $\Gamma(1/3)$ = the gamma function of 1/3 = 1.3541
- δ = parameter in the Pohlhausen profile, Equation (21)
- η = defined by Equation (15)
- θ = dimensionless group defined by Equation (6)
- ρ = density
- τ_{xy} = the shear stress
- τ_o = the shear stress at the wall
- ϕ = dimensionless group defined by Equation (6)

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